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Quasi-linear Stokes phenomenon for the Painlevé first equation

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Abstract

Using the Riemann–Hilbert approach, the Ψ-function corresponding to the solution of the first Painlevé equation $y_{xx} = 6y^2 + x$ with the asymptotic behaviour $y \sim \pm\sqrt{-x/6}$ as $|x| \to \infty$ is constructed. The exponentially small jump in the dominant solution and the coefficient asymptotics in the power-like expansion to the latter are found.

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1. Introduction

The Painlevé first equation [1]

$$y_{xx} = 6y^2 + x, \qquad (P_1)$$

is the simplest of the six classical equations of Painlevé–Gambier [2] and can be derived from any other Painlevé equation using certain scaling reductions [3]. Recent interest in this equation is due to its significant role in various physical models.

For instance, equation P_1 describes certain solutions to KdV and Bussinesq equations [4, 5], bifurcations in some non-integrable nonlinear models [6] and continuous limits in matrix models of quantum gravity [7–10]; the Ψ-function associated with P_1 appears in n-large asymptotics of semiclassical orthogonal and bi-orthogonal polynomials [11, 12] and thus becomes a primary object in the problem of Laplacian growth [13].

In the context of string theory, 'physical' solutions of P_1 are distinguished from 'non-physical' ones by the *monotonic* asymptotic behaviour as $x \to -\infty$ [7, 8, 10, 14, 15]. There are two kinds of such monotonic boundary conditions, i.e. $y(x) \simeq \pm\sqrt{-x/6}$. Using elementary perturbation analysis, the solution $y(x) = -\sqrt{-x/6} + \mathcal{O}(x^{-2})$ is unique as being a background to a two-parametric family of oscillating solutions. Solutions approaching a positive branch of the square root as $x \to -\infty$, i.e. $y(x) \simeq \sqrt{-x/6}$, form a one-parametric family parametrized by an amplitude from an exponentially small perturbation to a power-like dominant solution.

These solutions have the asymptotic expansion $y(x) = \sqrt{-x/6}\sum_{k=0}^{\infty} a_k(-x)^{-5k/2} + \mathcal{O}(x^{-\infty})$, whose coefficients a_k admit a combinatorial interpretation [16, 17].

In the problem of Laplacian growth without surface tension (Hele-Shaw problem, quantum Hall effect, etc), the shape of a growing droplet is described using a certain Ψ-function, see [13]. If the droplet develops a cusp singularity, this Ψ-function can be approximated by a Ψ-function associated with the first Painlevé transcendent [18]. Certain 'physical' asymptotic conditions imposed on this Ψ-function determine the relevant Stokes multipliers s_k. In turn, these s_k pinch out the monotonic as $x \to -\infty$ Painlevé function, $y(x) \simeq \sqrt{-x/6}$.

Equation P_1 has unexpectedly rich asymptotic properties in the complex x-plane. Boutroux [19] has shown that, generically, the asymptotics of the Painlevé first transcendent as $|x| \to \infty$ is described by the modulated Weierstrass elliptic function, whose module is a transcendent function of $\arg x$. Furthermore, the module function is such that the elliptic asymptotic ansatz degenerates along the directions $\arg x = \pi + (2\pi/5)n$, $n = 0, \pm 1, \pm 2$. Boutroux [19] called the corresponding trigonometric expansions *'truncated' solutions*. Their one- and zero-parameter reductions, if they admit analytic continuation into one or two neighbouring sectors of the complex x-plane, were called by Boutroux *'bi-truncated'* and *'tri-truncated' solutions*. All bi- and tri-truncated solutions have the algebraic leading order behaviour, $y(x) \sim \pm\sqrt{-x/6}$, perturbed by exponential terms.

We call a discontinuity in the asymptotic form of an analytic function the *Stokes phenomenon*. In the case of P_1, a jump in the phase shift of a modulated elliptic ansatz across the rays, $\arg x = \pi + (2\pi/5)n$, is called the *nonlinear* Stokes phenomenon. For bi- and tri-truncated solutions, a jump in the exponentially small perturbation of a dominant solution resembles the well-known Stokes phenomenon in the linear theory and is thus called the *quasi-linear* Stokes phenomenon.

In [20–22], equation P_1 was studied further using classical tools like the perturbation approach and the method of nonlinear integral equations. Mainly, these articles discuss the behaviour of the Painlevé transcendents on the real line. A recent paper [23] adopts the same approach, carefully studying the behaviour of the tri-truncated solution on the negative part of the real line.

In [24, 25], the multiple scale analysis was applied to P_1 (and P_2) to find a precise form of the phase shift in the elliptic asymptotic ansatz within complex sectors between the indicated rays. In [14, 15], the Witham averaging method was used to describe the elliptic tail of the monotonic at $-\infty$ solution of P_1.

The isomonodromy deformation approach to Painlevé equations, see [26–28], was applied to equation P_1 in [29–31]. In this way, the asymptotics of the Painlevé functions is expressed in terms of the Stokes multipliers of an associated linear system. Then, the equation of a monodromy surface yields connection formulae for the asymptotic parameters along different directions of the complex x-plane. A complete description of the nonlinear Stokes phenomenon in P_1 is given in [30]. A heuristic description of the quasi-linear Stokes phenomenon in P_1 can be found in [29].

Using the Borel transform technique and some assumptions on the analytic properties of the relevant Borel transforms, as well as the isomonodromy deformation approach based on the so-called exact WKB analysis, Takei [32] has re-derived the latter result (see [33] for more discussion).

In the present paper, we construct the Ψ-function associated with the monotonic as $|x| \to \infty$ solution of P_1 and rigorously describe the relevant quasi-linear Stokes phenomenon. Our main tool is the Riemann–Hilbert (RH) problem. We observe that the jump graph for our RH problem can be decomposed into a disjoint union of two branches, one of which is responsible for the background $\sqrt{-x/6}$, while the other produces the exponentially small perturbation of the dominant solution (see [33] for a similar observation in the P_2 case). Using the steepest-descent approach of Deift and Zhou [34], we prove the unique solubility of this problem and compute the asymptotics of the Painlevé transcendent.

Applying a rotational symmetry, we prove the existence of five solutions, $y_{4n}(x)$, $n = 0, \pm 1, \pm 2$, asymptotic to $\sqrt{e^{-i\pi}x/6}$ as $|x| \to \infty$ in the respective overlapping sectors $\arg x \in (-\frac{\pi}{5} - \frac{4\pi}{5}n, \frac{7\pi}{5} - \frac{4\pi}{5}n)$, see (2.69) and (2.72), and we find the exponentially small differences $y_{4(n-1)}(x) - y_{4n}(x)$, $n = 0, \pm 1, \pm 2$, see (2.71). The latter constitute the quasi-linear Stokes phenomenon.

A collection of the functions $y_{4n}(x)$, $n = 0, \pm 1, \pm 2$, forms a piecewise meromorphic function $\hat{y}(x) \sim \sqrt{e^{-i\pi}x/6}$ as $|x| \to \infty$. The moments of this function immediately yield the asymptotics as $k \to \infty$ for the coefficients a_k (3.12) of the x-series expansion to the dominant solution (3.1).

For the first time, the formula for the coefficient asymptotics was reported in [32]. In [23], the recurrence relations for the same coefficients were studied by direct means and it proves a similar asymptotic formula modulo a common factor. (An advanced version of the direct approach to a generalized recurrence relation, which contains one for P_1 as a special case, can be found in [35].) The exact value of this common factor was announced in [23] with reference to the method of [36] based on the Borel transform formula. In contrast, we do not use the Borel transform technique at any stage of our investigation.

This paper is organized as follows. In section 2, we recall the Lax pair for P_1, formulate the relevant RH problem and solve it asymptotically. Using the approximate Ψ-function, we find the asymptotics of the bi- and tri-truncated Painlevé transcendents and of the relevant Hamiltonian functions. In section 3, we find the coefficient asymptotics in the power-like expansion to the formal solution of P_1.

2. The RH problem for P_1

Introduce generators of $su(2, \mathbb{C})$,

$$\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad \sigma_+ = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}$$

and the Pauli matrices $\sigma_1 = \sigma_+ + \sigma_-$ and $\sigma_2 = -i\sigma_+ + i\sigma_-$ together and consider the system of matrix differential equations for Ψ (see [37, 38]):

$$\frac{\partial \Psi}{\partial \lambda}\Psi^{-1} = A(\lambda, x) = -z\sigma_3 + (2\lambda^2 + 2y\lambda + x + 2y^2)\sigma_+ + 2(\lambda - y)\sigma_-, \tag{2.1a}$$

$$\frac{\partial \Psi}{\partial x}\Psi^{-1} = U(\lambda, x) = (\lambda + 2y)\sigma_+ + \sigma_-. \tag{2.1b}$$

Compatibility of (2.1a) and (2.1b) implies that the coefficients z and y depend on the deformation parameter x in accordance with the nonlinear differential system

$$y_x = z, \qquad z_x = 6y^2 + x, \tag{2.2}$$

which is equivalent to the classical Painlevé first equation P_1. Following [26], see also [39], the linear equation (2.1a) has only the one (irregular) singular point at infinity, and there exist solutions $\Psi_k(\lambda)$ of (2.1) with the asymptotics

$$\Psi_k(\lambda) = \lambda^{\sigma_3/4}\frac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)(I - \mathcal{H}\sigma_3\lambda^{-1/2} + \mathcal{O}(\lambda^{-1}))e^{\theta(\lambda)\sigma_3}, \qquad \theta(\lambda) = \tfrac{4}{5}\lambda^{5/2} + x\lambda^{1/2}, \tag{2.3}$$

as

$$\lambda \to \infty, \qquad \lambda \in \Omega_k = \{\lambda \in \mathbb{C}\colon \arg\lambda \in (\tfrac{2\pi}{5}(k - \tfrac{3}{2}), \tfrac{2\pi}{5}(k + \tfrac{1}{2}))\}, \qquad k \in \mathbb{Z}. \tag{2.4}$$

Solutions $\Psi_k(\lambda)$, $k \in \mathbb{Z}$, are called the *canonical* solutions, while sectors Ω_k are called the *canonical* sectors. Canonical solutions $\Psi_k(\lambda)$ are uniquely determined by (2.3)–(2.4) and solve both the equations (2.1). They differ from each other in constant right matrix multipliers S_k called *Stokes matrices*,

$$\Psi_{k+1}(\lambda) = \Psi_k(\lambda)S_k, \qquad S_{2k-1} = \begin{pmatrix} 1 & s_{2k-1} \\ 0 & 1 \end{pmatrix}, \qquad S_{2k} = \begin{pmatrix} 1 & 0 \\ s_{2k} & 0 \end{pmatrix}. \tag{2.5}$$

Observing that all solutions of (2.1a) are entire functions, thus

$$\Psi_k(e^{2\pi i}\lambda) = \Psi_k(\lambda), \tag{2.6}$$

and using the relation

$$\Psi_{k+5}(e^{2\pi i}\lambda) = \Psi_k(\lambda)i\sigma_1, \tag{2.7}$$

which follows from the definition of the canonical solutions and the asymptotics (2.3) and (2.4), we readily find the constraints for the Stokes matrices [29],

$$S_{k+5} = \sigma_1 S_k \sigma_1, \qquad S_1S_2S_3S_4S_5 = i\sigma_1 \tag{2.8}$$

or, in scalar form,

$$s_{k+5} = s_k, \qquad 1 + s_k s_{k+1} = -i s_{k+3}, \qquad k \in \mathbb{Z}. \tag{2.8$'$}$$

Thus, generically, two of the Stokes multipliers s_k, $k \in \mathbb{Z}$, determine all others.

The inverse monodromy problem consists of reconstructing $\Psi_k(\lambda)$ using known values of the Stokes multipliers s_k. It can be equivalently formulated as an RH problem. With this aim, introduce the union of rays $\gamma = \rho \cup (\cup_{k=1}^{5}\gamma_{k-3})$, where $\gamma_k = \{\lambda \in \mathbb{C}\colon \arg\lambda = (2\pi/5)k\}$, $k = -2, -1, 0, 1, 2$, and $\rho = \{\lambda \in \mathbb{C} : \arg\lambda = \pi\}$, all oriented towards infinity. Denote the sectors between the rays ρ and γ_{-2} by ω_{-2}, those between γ_{k-1} and γ_k, $k = -1, 0, 1, 2$, by ω_k and those between γ_2 and ρ by ω_3. All the sectors ω_k are in one-to-one correspondence to the canonical sectors Ω_k (2.4), see figure 1.

Let each of the sectors ω_k, $k = -2, -1, \ldots, 3$, be a domain for a holomorphic 2×2 matrix function $\Psi_k(\lambda)$. Denote the collection of $\Psi_k(\lambda)$ by $\Psi(\lambda)$,

$$\Psi(\lambda)|_{\lambda\in\omega_k} = \Psi_k(\lambda). \tag{2.9}$$

Let $\Psi_+(\lambda)$ and $\Psi_-(\lambda)$ be the limits of $\Psi(\lambda)$ on γ to the left and the right, respectively.

Let $\theta(\lambda) = \frac{4}{5}\lambda^{5/2} + x\lambda^{1/2}$ be defined on the complex λ-plane cut along the negative part of the real axis. The RH problem we talk about is the following one:

1. find a piecewise holomorphic 2×2 matrix function $\Psi(\lambda)$ such that

   $$\lim_{\lambda\to\infty} \lambda^{1/2}(\tfrac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)\lambda^{-\sigma_3/4}\Psi(\lambda)e^{-\theta\sigma_3} - I) \tag{2.10}$$

   exists and is diagonal;
2. on the contour γ, the jump condition holds

   $$\Psi_+(\lambda) = \Psi_-(\lambda)S(\lambda), \tag{2.11}$$

   where the piecewise constant matrix $S(\lambda)$ is given by the following equations:

   $$S(\lambda)|_{\gamma_k} = S_k, \qquad S_{2k-1} = I + s_{2k-1}\sigma_+, \qquad S_{2k} = I + s_{2k}\sigma_-, \tag{2.12a}$$

   $$S(\lambda)|_{\rho} = -i\sigma_1, \tag{2.12b}$$

   with the constants s_k satisfying the constraints (2.8$'$).

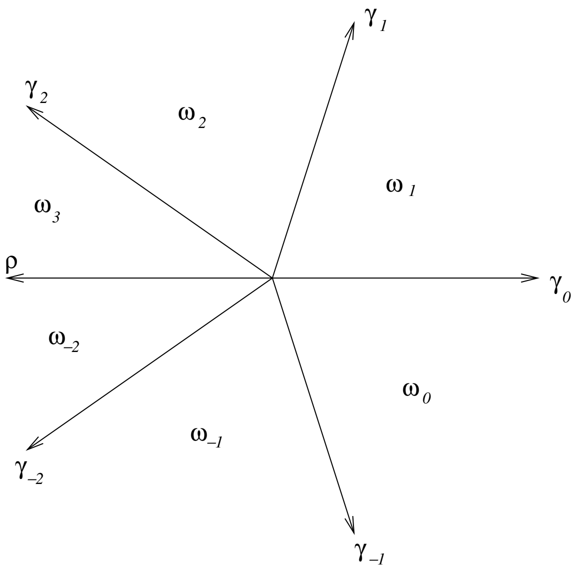

Figure 1. The RH problem graph for P_1.

Because $\Psi(\lambda)$ satisfies the asymptotic condition

$$\begin{aligned} Y(\lambda) &:= \frac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\lambda^{-\sigma_3/4}\Psi(\lambda)\mathrm{e}^{-\theta\sigma_3} \\ &= \begin{pmatrix} 1-(\mathcal{H}/\lambda^{1/2})+(\mathcal{H}^2/2\lambda)+\mathcal{O}(\lambda^{-3/2}) & (y/2\lambda)+\mathcal{O}(\lambda^{-3/2}) \\ (y/2\lambda)+\mathcal{O}(\lambda^{-3/2}) & 1+(\mathcal{H}/\lambda^{1/2})+(\mathcal{H}^2/2\lambda)+\mathcal{O}(\lambda^{-3/2}) \end{pmatrix}, \\ &\quad \lambda\to\infty, \end{aligned} \tag{2.13}$$

where

$$\mathcal{H} = \tfrac{1}{2}z^2 - 2y^3 - xy, \tag{2.14}$$

the solution $y(x)$ of the Painlevé equation can be found from the 'residue' of $Y(\lambda)$ at infinity,

$$y = 2\lim_{\lambda\to\infty}\lambda Y_{12}(\lambda) = 2\lim_{\lambda\to\infty}\lambda Y_{21}(\lambda). \tag{2.15}$$

Remark 2.1. It is easy to see that $\mathcal{H}$ is nothing but the Hamiltonian for the Painlevé first equation with the canonical variables $q = y$ and $p = z$.

Equation (2.15) specifies the Painlevé transcendent as a function $y = f(x, \{s_k\})$ of the deformation parameter x and of the Stokes multipliers s_k. Using the solution $y = f(x, \{s_k\})$ and the symmetries of the Stokes multipliers described in [29], we obtain further solutions of P_1:

$$y = \overline{f(\bar{x}, \{-\overline{s_{-k}}\})}, \tag{2.16a}$$

$$y = \mathrm{e}^{\mathrm{i}(4\pi/5)n} f(\mathrm{e}^{\mathrm{i}(2\pi/5)n}x, \{s_{k+2n}\}), \qquad n\in\mathbb{Z}, \tag{2.16b}$$

where the over bar implies complex conjugation.

For technical reasons, to find the asymptotics of $y(x)$, we use below not $Y(\lambda)$ but the related auxiliary functions $\chi(\lambda)$ and $X(\lambda)$ with expansions (2.45) and (2.50), respectively. The latter involve differences $y - \hat{y}(x)$, $\hat{y}(x)$ are known, which can be estimated using singular integral equations with contracting operators.

2.1. Asymptotic solution for $s_0 = 0$

Let us consider the RH problem above, where $s_0 = 0$ assuming that $|x| \to \infty$ within the sector $\arg x \in [(3\pi/5), \pi]$. Equations (2.12a) imply that $\Psi(\lambda)$ has no jump across the ray $\gamma_0 = \{\lambda \in \mathbb{C} : \arg\lambda = 0\}$. The constraints (2.8′) reduce to the following system of equations:

$$s_{-2} = s_2 = s_{-1} + s_1 = \mathrm{i}. \tag{2.17}$$

Our first step in the RH problem analysis consists of introducing an auxiliary g-function,

$$g(\lambda) = \tfrac{4}{5}(\lambda + 2\lambda_0)^{5/2} - 4\lambda_0(\lambda + 2\lambda_0)^{3/2}, \qquad \lambda_0 = \sqrt{\mathrm{e}^{-\mathrm{i}\pi}x/6}, \tag{2.18}$$

defined on the complex λ-plane cut along the ray $(-\infty, -2\lambda_0]$. The asymptotics of the g-function as $\lambda \to \infty$ coincides with the canonical one,

$$g(\lambda) = \tfrac{4}{5}\lambda^{5/2} - 6\lambda_0^2\lambda^{1/2} - 4\lambda_0^3\lambda^{-1/2} + \mathcal{O}(\lambda^{-3/2}) = \tfrac{4}{5}\lambda^{5/2} + x\lambda^{1/2} + \mathcal{O}(\lambda^{-1/2}). \tag{2.19}$$

Let us formulate an equivalent RH problem for the piecewise holomorphic function $Z(\lambda)$,

$$Z(\lambda) = Y(\lambda)\mathrm{e}^{(\theta(\lambda)-g(\lambda))\sigma_3} = \tfrac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)\lambda^{-\sigma_3/4}\Psi(\lambda)\mathrm{e}^{-g(\lambda)\sigma_3}, \tag{2.20}$$

(i) $Z(\lambda) \to I$ as $\lambda \to \infty$;

(ii) $Z_+(\lambda) = Z_-(\lambda)G(\lambda)$, $\quad G(\lambda) = \mathrm{e}^{g\sigma_3}S(\lambda)\mathrm{e}^{-g\sigma_3}$, $\quad \lambda \in \gamma_k$, (2.21)

$Z_+(\lambda) = \sigma_1 Z_-(\lambda)\sigma_1$, $\quad \lambda \in \rho$.

If $S(\lambda) = I + s\sigma_\pm$ then $G(\lambda) = I + s\mathrm{e}^{\pm 2g}\sigma_\pm$. Our next goal is to transform the jump contour γ to the contour of the steepest descent for the matrix $G(\lambda) - I$. We denote by γ_+ the level line $\operatorname{Im} g(\lambda) = \text{const}$ passing through the stationary phase point $\lambda = \lambda_0 = \sqrt{\mathrm{e}^{-\mathrm{i}\pi}x/6}$ and asymptotic to the rays $\arg\lambda = \pm 2\pi/5$. This is the steepest descent line for e^{2g}. Let $\gamma_- = \cup_j \ell_j \cup \sigma$ be the union of the level lines ℓ_j, $j = 0, 1, 2$, $\operatorname{Im} g(\lambda) = \text{const}$, and σ, $\operatorname{Re} g(\lambda) = \text{const}$, all emanating from the critical point $\lambda = -2\lambda_0$. Among them, the level line ℓ_1 approaching the ray $\arg\lambda = 2\pi/5$ (if $\arg x = \pi$, the level line ℓ_1 is the segment $[-2\lambda_0, \lambda_0]$) is the steepest descent line for e^{2g}, while the level lines ℓ_0 and ℓ_2 approaching the rays $\arg\lambda = -4\pi/5$ and $\arg\lambda = 4\pi/5$, respectively, are the steepest descent lines for e^{-2g}. The level line σ, $\operatorname{Re} g(\lambda) = \text{const}$, approaches the ray $\arg\lambda = \pi$.

Since the Stokes matrix S_1 can be factorized,

$$S_1 = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{i} - s_{-1} \\ 0 & 1 \end{pmatrix} = S_{-1}^{-1}\begin{pmatrix} 1 & \mathrm{i} \\ 0 & 1 \end{pmatrix},$$

it is convenient to consider the following equivalent RH problem for $\Psi(\lambda)$.

For $\arg x \in [3\pi/5, \pi]$, the jump contour is the union of γ_+ oriented from up to down and γ_- whose components are oriented towards infinity, see figure 2. The jump matrices

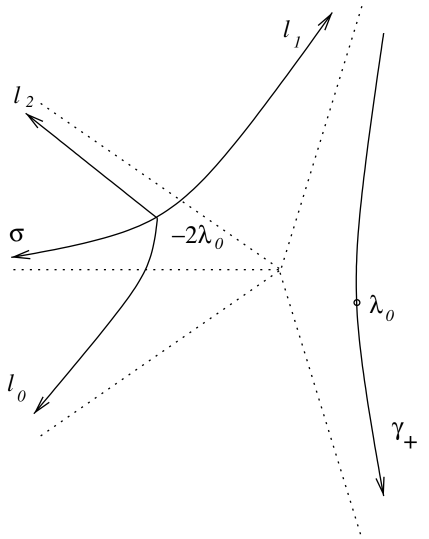

Figure 2. An RH problem graph for $s_0 = 0$.

are as follows:

$$\begin{aligned}
&\lambda \in \gamma_+ : && S(\lambda) = S_{-1} = \begin{pmatrix} 1 & s_{-1} \\ 0 & 1 \end{pmatrix}, \\
&\lambda \in \ell_1 : && S(\lambda) = S_+ = \begin{pmatrix} 1 & \mathrm{i} \\ 0 & 1 \end{pmatrix}, \\
&\lambda \in \ell_0 \cup \ell_2 : && S(\lambda) = S_- = \begin{pmatrix} 1 & 0 \\ \mathrm{i} & 1 \end{pmatrix}, \\
&\lambda \in \sigma : && S(\lambda) = -\mathrm{i}\sigma_1.
\end{aligned} \tag{2.22}$$

Remark 2.2. The jump contour for the RH problem (2.22) is decomposed into the disjoint union of the line γ_+ and the graph γ_-, see figure 2. For the boundary value $\arg x = \pi - 0$, the level line ℓ_1 emanating from $\lambda = -2\lambda_0$ passes through the stationary phase point $\lambda = \lambda_0$ and partially merges with the upper half of the level line γ_+. To construct the RH problem however, it is not necessary to transform the jump contour precisely to the steepest descent graph. It is enough to ensure that the jump matrices approach the unit matrix uniformly with respect to λ and fast enough as $x \to \infty$ or $\lambda \to \infty$.

As $\arg x \in [3\pi/5, \pi]$, introduce the reduced RH problem ($s_0 = s_{-1} = 0$) for the piecewise holomorphic function $\Phi(\lambda)$ discontinuous across γ_- only:

$$\begin{aligned}
&\text{(i) } \lim_{\lambda\to\infty} \lambda^{1/2}\Big(\tfrac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\lambda^{-\frac{1}{4}\sigma_3}\Phi(\lambda)\mathrm{e}^{-\theta\sigma_3} - I\Big) \quad \text{is diagonal,} \\
&\text{(ii) } \Phi_+(\lambda) = \Phi_-(\lambda)S(\lambda), \quad \lambda \in \gamma_- = \bigcup_{j=0,1,2} \ell_j \bigcup \sigma.
\end{aligned} \tag{2.23}$$

The jump matrix $S(\lambda)$ here is defined in (2.22).

Theorem 2.1. *If* $\arg x \in [3\pi/5, \pi]$ *and* $|x|$ *is large enough, then there exists a unique solution of the RH problem* (2.23). *The Painlevé function* $y_0(x)$ *corresponding to* $s_0 = s_{-1} = 0$ *has the asymptotics* $y_0(x) = \sqrt{\mathrm{e}^{-\mathrm{i}\pi}x/6} + \mathcal{O}(x^{-2})$ *as* $x \to \infty$ *in the above sector.*

Proof. *Uniqueness.* Since $\det S(\lambda) \equiv 1$, we have $\det \Phi_+ = \det \Phi_-$, and hence $\det \Phi(\lambda)$ is an entire function. Furthermore, because of normalization of $\Phi(\lambda)$ at infinity, $\det \Phi(\lambda) \equiv -1$. Let $\tilde{\Phi}$ and Φ be two solutions of (2.23). Taking into account the cyclic relation in (2.8) which implies the continuity of the RH problem for $\Phi(\lambda)$ at $\lambda = -2\lambda_0$, the 'ratio' $\chi(\lambda) = \tilde{\Phi}(\lambda)\Phi^{-1}(\lambda)$ is an entire function of λ. Using the Liouville theorem and normalization of Φ and $\tilde{\Phi}$ at infinity, we find $\chi(\lambda) \equiv I$, i.e. $\Phi(\lambda) \equiv \tilde{\Phi}(\lambda)$.

Existence. Introduce an auxiliary function

$$\hat{\Phi}_0(z) = \begin{pmatrix} v_1'(z) & v_2'(z) \\ v_1(z) & v_2(z) \end{pmatrix}, \tag{2.24}$$

where the prime means differentiation w.r.t. z and

$$v_1(z) = \sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi/6}\mathrm{Ai}(\mathrm{e}^{\mathrm{i}2\pi/3}z), \qquad v_2(z) = -\sqrt{2\pi}\mathrm{Ai}(z), \tag{2.25}$$

with $\mathrm{Ai}(z)$ standing for the classical Airy function, which can be defined using the Taylor expansion [40, 41],

$$\mathrm{Ai}(z) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})}\sum_{k=0}^{\infty}\frac{3^k\Gamma(k+\frac{1}{3})z^{3k}}{\Gamma(\frac{1}{3})(3k)!} - \frac{1}{3^{1/3}\Gamma(\frac{1}{3})}\sum_{k=0}^{\infty}\frac{3^k\Gamma(k+\frac{2}{3})z^{3k+1}}{\Gamma(\frac{2}{3})(3k+1)!}. \tag{2.26}$$

Asymptotics at infinity of this function and its derivative are as follows:

$$\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi}}z^{-1/4}\exp(-\tfrac{2}{3}z^{3/2})\left\{\sum_{n=0}^{N}(-1)^n 3^{-2n}\frac{\Gamma(3n+\frac{1}{2})}{\Gamma\frac{1}{2}(2n)!}z^{-3n/2} + \mathcal{O}(z^{-3(N+1)/2})\right\},$$

$$\mathrm{Ai}'(z) = \frac{1}{2\sqrt{\pi}}z^{1/4}\exp\left(-\tfrac{2}{3}z^{3/2}\right)\left\{\sum_{n=0}^{N}(-1)^n 3^{-2n}(3n+\tfrac{1}{2})\frac{\Gamma(3n-\frac{1}{2})}{\Gamma\frac{1}{2}(2n)!}z^{-3n/2} \right. \tag{2.27}$$

$$\left. + \mathcal{O}(z^{-3(N+1)/2})\right\}, \qquad \text{as} \quad z \to \infty, \quad \arg z \in (-\pi, \pi).$$

It is worth noting that the function $\hat{\Phi}_0(z)$ satisfies the linear differential equation

$$\frac{\mathrm{d}\hat{\Phi}_0}{\mathrm{d}z} = \{z\sigma_+ + \sigma_-\}\hat{\Phi}_0. \tag{2.28}$$

Using the properties of the Airy functions, we find that the products

$$\hat{\Phi}_1(z) = \hat{\Phi}_0(z)S_-, \qquad \hat{\Phi}_2(z) = \hat{\Phi}_1(z)S_+, \qquad \hat{\Phi}_3(z) = \hat{\Phi}_2(z)S_-, \tag{2.29}$$

where $S_\pm = I + \mathrm{i}\sigma_\pm$, have the asymptotic expansion

$$\hat{\Phi}_k(z) = z^{\sigma_3/4}\tfrac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)V_\infty(z)\exp(\tfrac{2}{3}z^{3/2}\sigma_3), \tag{2.30}$$

as $|z| \to \infty$, $\arg z \in (-\pi + (2\pi/3)k, (\pi/3) + (2\pi/3)k)$, where

$$V_\infty(z) = I - \sum_{n=1}^{\infty}3^{-2n}\frac{\Gamma(3n-\frac{1}{2})}{2\Gamma\frac{1}{2}(2n)!}z^{-3n/2}\begin{pmatrix} 1 & (-1)^n 6n \\ 6n & (-1)^n \end{pmatrix}. \tag{2.31}$$

Let $\hat{\gamma}_- = \hat{\sigma} \cup_{j=0,1,2} \hat{\ell}_j$ be the union of the rays $\hat{\ell}_j = \{z \in \mathbb{C} : \arg z = (2\pi/3)(j-1)\}$, $j = 0, 1, 2$, and $\hat{\sigma} = \{z \in \mathbb{C} : \arg z = \pi\}$, all oriented towards infinity. This graph divides the

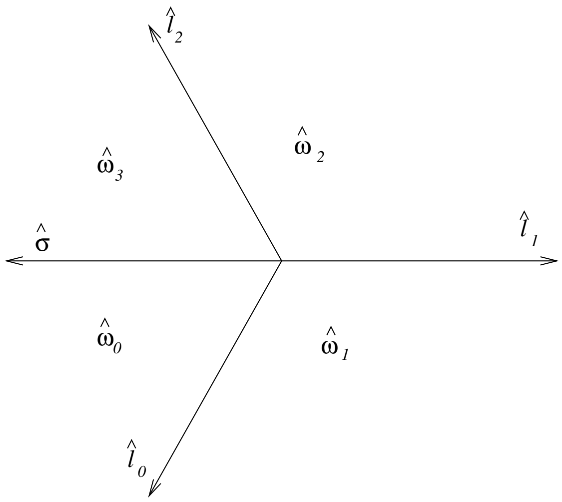

Figure 3. The model RH problem graph.

complex z-plane into four regions: the sector $\hat{\omega}_0$ between $\hat{\sigma}$ and $\hat{\ell}_0$; the sectors $\hat{\omega}_k$, $k = 1, 2$, between the rays $\hat{\ell}_{k-1}$ and $\hat{\ell}_k$; and the sector $\hat{\omega}_3$ between the rays $\hat{\ell}_2$ and $\hat{\sigma}$.

Define a piecewise holomorphic function $\hat{\Phi}(z)$,

$$\hat{\Phi}(z)|_{z\in\hat{\omega}_k} = \hat{\Phi}_k(z). \tag{2.32}$$

By construction, this function solves the following RH problem (see figure 3):

(i) $\frac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)z^{-\sigma_3/4}\hat{\Phi}(z)\exp\left(-\frac{2}{3}z^{3/2}\sigma_3\right) = I + \mathcal{O}(z^{-3/2}), \quad z \to \infty,$ (2.33)

(ii)
$$\begin{aligned} &z \in \hat{\gamma}_- : \quad \hat{\Phi}_+(z) = \hat{\Phi}_-(z)\hat{S}(z),\\ &z \in \hat{\ell}_1 : \quad \hat{S}(z) = S_+, \quad z \in \hat{\ell}_0 \cup \hat{\ell}_2 : \quad \hat{S}(z) = S_-,\\ &z \in \hat{\sigma} : \quad \hat{S}(z) = -\mathrm{i}\sigma_1. \end{aligned} \tag{2.34}$$

Therefore the function $\hat{\Phi}(z)$ has precisely the jump properties of the function $\Phi(\lambda)$. To find $\Phi(\lambda)$ with the correct asymptotic behaviour at infinity, let us use the mapping

$$\tfrac{2}{3}z^{3/2} = g(\lambda) = \tfrac{4}{5}(\lambda + 2\lambda_0)^{5/2} - 4\lambda_0(\lambda + 2\lambda_0)^{3/2}$$

or

$$z(\lambda) = (-6\lambda_0)^{2/3}(\lambda + 2\lambda_0)\left(1 - \frac{1}{5\lambda_0}(\lambda + 2\lambda_0)\right)^{2/3}, \qquad \lambda_0 = \sqrt{\mathrm{e}^{-\mathrm{i}\pi}x/6}, \tag{2.35}$$

Within the disc $|\lambda + 2\lambda_0| \leqslant R < 3|\lambda_0| = |\frac{3}{2}x|^{1/2}$, the mapping (2.35) yields a holomorphic change of the independent variable. Introduce a piecewise holomorphic function $\tilde{\Phi}(\lambda)$,

$$\tilde{\Phi}(\lambda) = \begin{cases} B(\lambda)\hat{\Phi}(z(\lambda)), & |\lambda + 2\lambda_0| < R,\\ (\lambda + 2\lambda_0)^{\sigma_3/4}\frac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)\mathrm{e}^{g(\lambda)\sigma_3}, & |\lambda + 2\lambda_0| > R, \end{cases}$$

$$B(\lambda) = (-6\lambda_0)^{-\sigma_3/6}\left(1 - \frac{\lambda + 2\lambda_0}{5\lambda_0}\right)^{-\sigma_3/6}, \tag{2.36}$$

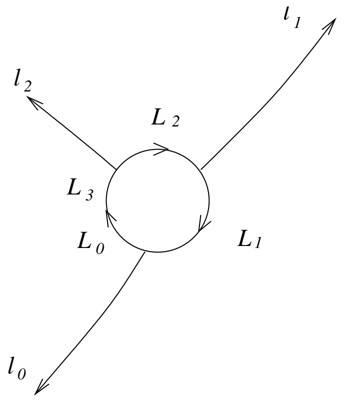

Figure 4. An RH problem graph for the correction function $\chi(\lambda)$.

where $(\lambda + 2\lambda_0)^{1/4}$ is defined on the plane cut along the level line σ asymptotic to the ray $\arg\lambda = \pi$. Note that $B(\lambda)$ is holomorphic in the interior of the above disc $|\lambda+2\lambda_0| \leqslant R < 3|\lambda_0|$ and thus does not affect the jump properties of $\hat{\Phi}(z(\lambda))$. We look for the solution of the RH problem (2.23) in the form of the product

$$\Phi(\lambda) = (I + (4\lambda_0^3 - \mathcal{H})\sigma_+)\chi(\lambda)\tilde{\Phi}(\lambda). \tag{2.37}$$

Consider the RH problem for the correction function $\chi(\lambda)$. By construction, it is a piecewise holomorphic function discontinuous across the clockwise-oriented circle $\mathcal{L}$ of radius R centred at $-2\lambda_0$ and across the part of γ_- located outside the above circle. (In fact, $\chi(\lambda)$ is continuous across σ, see (2.39) below.) The latter is divided by γ_- into four arcs: $\mathcal{L}_0$ between σ and ℓ_0; $\mathcal{L}_k, k = 1, 2$, between ℓ_{k-1} and ℓ_k; and $\mathcal{L}_3$ between ℓ_2 and σ, see figure 4. To simplify our notation, let us put

$$\tilde{\lambda} = \lambda + 2\lambda_0. \tag{2.38}$$

Then the RH problem for $\chi(\lambda)$ is as follows:

(i) $\chi(\lambda) \to I, \qquad \lambda \to \infty;$

(ii) $\chi^+(\lambda) = \chi^-(\lambda)\mathcal{G}(\lambda), \qquad \lambda \in \ell,$

where

$$\begin{aligned} &\lambda \in \ell_1,\ |\tilde{\lambda}| > R: && \mathcal{G}(\lambda) = I + \tfrac{\mathrm{i}}{2}\mathrm{e}^{2g}(\sigma_3 - \tilde{\lambda}^{1/2}\sigma_+ + \tilde{\lambda}^{-1/2}\sigma_-),\\ &\lambda \in \ell_0 \cup \ell_2, |\tilde{\lambda}| > R: && \mathcal{G}(\lambda) = I + \tfrac{\mathrm{i}}{2}\mathrm{e}^{-2g}(\sigma_3 + \tilde{\lambda}^{1/2}\sigma_+ - \tilde{\lambda}^{-1/2}\sigma_-),\\ &\lambda \in \sigma, |\tilde{\lambda}| > R: && \mathcal{G}(\lambda) = I,\\ &|\tilde{\lambda}| = R, \lambda \in \mathcal{L}_k: && \mathcal{G}(\lambda) = B(\lambda)\hat{\Phi}_k(z(\lambda))\mathrm{e}^{-g\sigma_3}\tfrac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)\tilde{\lambda}^{-\frac{1}{4}\sigma_3},\\ & && k = 0, 1, 2, 3. \end{aligned} \tag{2.39}$$

Taking into account the equations (2.34), it is easy to check the continuity of the RH problem at the node points. Observing that, on the circle $|\tilde{\lambda}| = R = c|x|^{1/2}$, $0 < c < \sqrt{3/2}$, we have that $z(\lambda) = \mathcal{O}(|x|^{5/6})$ is large, we immediately see that

$$\|\mathcal{G}(\lambda) - I\| \leqslant c|\tilde{\lambda}|^{1/2}\mathrm{e}^{-(2/3)^{1/2}|x|^{1/2}|\tilde{\lambda}|^{3/2}}, \qquad \lambda \in \ell_k, \quad k = 0, 1, 2, \quad |\tilde{\lambda}| \geqslant R, \tag{2.40}$$

where the precise value of the positive constant c is not important for us. Taking into account that, for the above reason, on the circle $|\tilde{\lambda}| = R$, we may use for $\hat{\Phi}_k$ its asymptotics (2.30), the jump matrix $\mathcal{G}(\lambda)$ has the asymptotic expansion

$$\mathcal{G}(\lambda) - I = -\sum_{n=1}^{\infty} 3^{-2n} \frac{\Gamma(3n - \frac{1}{2})}{2\Gamma(\frac{1}{2})(2n)!} z^{-\frac{3n}{2}}$$
$$\times \begin{pmatrix} \frac{1+(-1)^n}{2}(1+6n) & \frac{1-(-1)^n}{2}(1+6n)\tilde{\lambda}^{1/2} \\ \frac{1-(-1)^n}{2}(1-6n)\tilde{\lambda}^{-1/2} & \frac{1+(-1)^n}{2}(1-6n) \end{pmatrix},$$
$$z = (-6\lambda_0)^{2/3}\left(1 - \frac{\tilde{\lambda}}{5\lambda_0}\right)^{2/3}\tilde{\lambda}, \qquad \tilde{\lambda} = \lambda + 2\lambda_0, \quad |\tilde{\lambda}| = R. \tag{2.41}$$

Therefore, we have the estimate

$$\|\mathcal{G}(\lambda) - I\| \leqslant cR^{-2} = c'|x|^{-1}, \tag{2.42}$$

where the precise values of the positive constants c, c' are not important for us.

Now, the solvability of the RH problem (2.39), and therefore of (2.23), for large enough $|x|$ is straightforward. Indeed, consider the equivalent system of the non-homogeneous singular integral equations for the limiting value $\chi^+(\lambda)$, i.e.

$$\chi^+(\lambda) = I - \frac{1}{2\pi\mathrm{i}}\int_\ell \frac{\chi^+(\zeta)(\mathcal{G}^{-1}(\zeta) - I)}{\zeta - \lambda_+}\mathrm{d}\zeta, \tag{2.43}$$

or, in symbolic form, $\chi^+ = I + K\chi^+$. Here λ_+ means the left limit of λ on ℓ (recall that the circle $|\tilde{\lambda}| = R$ is clockwise oriented), and K is the composition of the operator of the right multiplication in $\mathcal{G}^{-1}(\lambda) - I$ and of the Cauchy operator C_+. An equivalent singular integral equation for $\psi^+ := \chi^+ - I$ differs from (2.43) in the inhomogeneous term only,

$$\psi^+ = KI + K\psi^+. \tag{2.44}$$

Consider the integral equation (2.44) in the space $L_2(\ell)$. Since $\mathcal{G}^{-1}(\lambda) - I$ is small in $L_2(\ell)$ for large enough $|x|$, and C_+ is bounded in $L_2(\ell)$, then $\|K\|_{L_2(\ell)} \leqslant c|x|^{-1/2}$ with some positive constant c, thus K is contracting and $I - K$ is invertible in $L_2(\ell)$ for large enough $|x|$. Because $KI \in L_2(\ell)$, equation (2.44) for ψ^+ is solvable in $L_2(\ell)$, and the solution $\chi(\lambda)$ of the RH problem (2.39) is determined by $\psi^+(\lambda)$ using the equation $\chi^+ = I + KI + K\psi^+$.

Let us find the asymptotics of the Painlevé function. Using (2.13) and the definition of $\tilde{\Phi}(\lambda)$ (2.36), the asymptotics of $\chi(\lambda)$ as $\lambda \to \infty$ in terms of y and $\mathcal{H}$ is

$$\chi(\lambda) = I + \frac{1}{2\lambda}(y - \lambda_0 - (4\lambda_0^3 - \mathcal{H})^2)\sigma_3 + \frac{1}{\lambda}(4\lambda_0^3 - \mathcal{H})\sigma_- + \begin{pmatrix} \mathcal{O}(\lambda^{-3/2}) & \mathcal{O}(\lambda^{-1}) \\ \mathcal{O}(\lambda^{-2}) & \mathcal{O}(\lambda^{-3/2}) \end{pmatrix}. \tag{2.45}$$

On the other hand, in accord with the above, the function χ^+ is given by the converging iterative series, $\chi^+ = \sum_{n=0}^{\infty} K^n I$. To compute the term $K^n I$, we observe that the contribution of the infinite branches ℓ_k is exponentially small in x due to estimate (2.40). Using the expansion (2.41), we reduce the evaluation of the integral along the circle $|\tilde{\lambda}| = R$ to the residue theorem. Omitting this elementary computation, we present the final result: for large enough $|x|$, $\arg x \in [\frac{3\pi}{5}, \pi]$, the asymptotics of $\chi(\lambda)$ as $\lambda \to \infty$ is given by

$$\chi(\lambda) = I + \begin{pmatrix} \mathcal{O}(\lambda_0^{-4}\tilde{\lambda}^{-1}) & \mathcal{O}(\lambda_0^{-1}\tilde{\lambda}^{-1}) \\ \mathcal{O}(\lambda_0^{-2}\tilde{\lambda}^{-1}) & \mathcal{O}(\lambda_0^{-4}\tilde{\lambda}^{-1}) \end{pmatrix}. \tag{2.46}$$

Comparing entries $\chi_{21}(\lambda)$ in (2.45) and (2.46), we see that the Hamiltonian function $\mathcal{H} = \mathcal{H}_0(x)$ corresponding to the Stokes multipliers $s_0 = s_{-1} = 0$ is given by

$$\mathcal{H} = \mathcal{H}_0(x) = 4\lambda_0^3 + \mathcal{O}(\lambda_0^{-2}) = 4(-x/6)^{3/2} + \mathcal{O}(x^{-1}). \tag{2.47}$$

Next, comparing entries $\chi_{11}(\lambda)$ in (2.45) and (2.46) and using (2.47), we find the asymptotics of the Painlevé function $y = y_0(x)$,

$$y = y_0(x) = \lambda_0 + \mathcal{O}(\lambda_0^{-4}) = \sqrt{-x/6} + \mathcal{O}(x^{-2}). \tag{2.48}$$

Recall that $\lambda_0 = (\mathrm{e}^{-\mathrm{i}\pi}x/6)^{1/2}$ where the main branch of the root is taken. □

Let us go to the case of the nontrivial s_{-1} described by the RH problem (2.22). We look for the solution $\Psi(\lambda)$ in the form of the product

$$\Psi(\lambda) = (I - (\mathcal{H} - \mathcal{H}_0)\sigma_+)X(\lambda)\Phi(\lambda), \tag{2.49}$$

where $\Phi(\lambda)$ is the solution of the reduced RH problem (2.23) and $\mathcal{H}_0$ (2.47) is the Hamiltonian function (2.14) corresponding to the Painlevé transcendent $y_0(x)$ (2.48). Using (2.13), we find the asymptotics of $X(\lambda)$ as $\lambda \to \infty$,

$$\begin{aligned} X(\lambda) &= (I + (\mathcal{H} - \mathcal{H}_0)\sigma_+)\Psi\Phi^{-1} \\ &= I + \frac{1}{2\lambda}(y - y_0 - (\mathcal{H} - \mathcal{H}_0)^2)\sigma_3 - \frac{1}{\lambda}(\mathcal{H} - \mathcal{H}_0)\sigma_- + \begin{pmatrix} \mathcal{O}(\lambda^{-3/2}) & \mathcal{O}(\lambda^{-1}) \\ \mathcal{O}(\lambda^{-2}) & \mathcal{O}(\lambda^{-3/2}) \end{pmatrix}. \end{aligned} \tag{2.50}$$

Thus we arrive at the RH problem for the correction function $X(\lambda)$ on the steepest descent line γ_+,

$$\begin{array}{lll} \text{(i)} & X(\lambda) \to I, & \lambda \to \infty, \\ \text{(ii)} & X_+(\lambda) = X_-(\lambda)\mathcal{G}(\lambda), & \lambda \in \gamma_+, \\ & \mathcal{G}(\lambda) = \Phi(\lambda)S_{-1}\Phi^{-1}(\lambda) & \end{array} \tag{2.51}$$

Note, $\Phi(\lambda)$ is continuous across γ_+ and therefore holomorphic in some neighbourhood of γ_+ as $\arg x \in [(3\pi/5), \pi]$, and $|x|$ is large enough.

The jump matrix on γ_+ can be estimated as follows:

$$\begin{aligned} \|\mathcal{G}(\lambda) - I\| &\leqslant c|s_{-1}| \exp\left[-\tfrac{1}{5}2^{11/4}3^{1/4}|x|^{5/4}\cos\left(\tfrac{5}{4}(\arg x - \pi)\right)\right] \\ &\quad \times \exp\left[-2^{3/4}3^{1/4}|x|^{1/4}|\lambda - \lambda_0|^2\right]. \end{aligned} \tag{2.52}$$

Here c is some positive constant whose precise value is not important for us and $\lambda_0 = (\mathrm{e}^{-\mathrm{i}\pi}x)^{1/2}$ is the stationary phase point for $\exp(g(\lambda))$, see (2.18). Estimate (2.52) yields the estimate for the norm of the singular integral operator $\mathcal{K}$ in the equivalent system of singular integral equations, $X_- = I + \mathcal{K}X_-$,

$$\|\mathcal{K}\|_{L_2(\gamma_+)} \leqslant c'|s_{-1}|\exp\left[-\tfrac{1}{5}2^{11/4}3^{1/4}|x|^{5/4}\cos\left(\tfrac{5}{4}(\arg x-\pi)\right)\right], \quad c'>0. \tag{2.53}$$

If $|x|$ is large enough and $\arg x \in [(3\pi/5)+\epsilon, \pi]$, $\epsilon > 0$, then the operator $\mathcal{K}$ is contracting and the system $X_- = I + \mathcal{K}X_-$ is solvable by iterations in $L_2(\gamma_+)$, i.e. $X_- = \sum_{n=0}^{\infty}\mathcal{K}^n X_-$. However, to incorporate the oscillating direction $\arg x = 3\pi/5$ in the general scheme, we use a more refined procedure.

Theorem 2.2. *If $s_0 = 0$, $\arg x \in [(3\pi/5), \pi]$ and $|x|$ is large enough, then there exists a unique solution of the RH problem* (2.10)–(2.12). *The corresponding Painlevé function has the asymptotics*

$$y(x) = y_0(x) + \frac{s_{-1}}{\sqrt{\pi}}2^{-11/8}3^{-1/8}(\mathrm{e}^{-\mathrm{i}\pi}x)^{-1/8}\exp\left[-\tfrac{1}{5}2^{11/4}3^{1/4}(\mathrm{e}^{-\mathrm{i}\pi}x)^{5/4}\right](1+\mathcal{O}(x^{-3/8})), \tag{2.54}$$

where $y_0(x) \sim \sqrt{\mathrm{e}^{-\mathrm{i}\pi}x/6}$ is the solution of the Painlevé equation for $s_0 = s_{-1} = 0$, $s_1 = s_2 = s_{-2} = \mathrm{i}$.

Proof. It is enough to prove the solvability of the RH problem (2.51).

Using, for $\Phi(\lambda)$, the expressions (2.37) with (2.36) and the estimate (2.46) together, we find the asymptotics of the jump matrix $\mathcal{G}(\lambda)$,

$$\mathcal{G}(\lambda) = I + \tfrac{1}{2}s_{-1}\mathrm{e}^{2g}\begin{pmatrix} 1+\mathcal{O}(\lambda_0^{-2}\tilde{\lambda}^{-1/2}) & -\tilde{\lambda}^{1/2}+\mathcal{O}(\lambda_0^{-2}) \\ \tilde{\lambda}^{-1/2}+\mathcal{O}(\lambda_0^{-2}\tilde{\lambda}^{-1}) & -1+\mathcal{O}(\lambda_0^{-2}\tilde{\lambda}^{-1/2}) \end{pmatrix}, \quad \lambda\in\gamma_+, \quad \tilde{\lambda} = \lambda + 2\lambda_0. \tag{2.55}$$

Consider the following model RH problem:

$$\begin{aligned} &\text{(i)} \quad P(\lambda) \to I, \qquad \lambda\to\infty, \\ &\text{(ii)} \quad P_+(\lambda) = P_-(\lambda)\hat{\mathcal{G}}(\lambda), \qquad \lambda\in\gamma_+, \\ &\qquad \hat{\mathcal{G}}(\lambda) = I + \tfrac{1}{2}s_{-1}\mathrm{e}^{2g}\begin{pmatrix} 1 & -(3\lambda_0)^{1/2} \\ (3\lambda_0)^{-1/2} & -1 \end{pmatrix}. \end{aligned} \tag{2.56}$$

This problem is solvable by the following quadrature:

$$P(\lambda) = I + \frac{1}{2}s_{-1}\frac{1}{2\pi\mathrm{i}}\int_{\gamma_+}\frac{\mathrm{e}^{2g}}{\zeta-\lambda}\mathrm{d}\zeta\begin{pmatrix} 1 & -(3\lambda_0)^{1/2} \\ (3\lambda_0)^{-1/2} & -1 \end{pmatrix}. \tag{2.57}$$

We look for the solution $X(\lambda)$ of the RH problem (2.51) in the form of the product,

$$X(\lambda) = Q(\lambda)P(\lambda). \tag{2.58}$$

The correction function $Q(\lambda)$ satisfies the RH problem

$$\begin{aligned} &\text{(i)} \quad Q(\lambda) \to I, \qquad \lambda\to\infty, \\ &\text{(ii)} \quad Q_+(\lambda) = Q_-(\lambda)W(\lambda), \qquad \lambda\in\gamma_+, \\ &\qquad W(\lambda) = P_-(\lambda)\mathcal{G}(\lambda)\hat{\mathcal{G}}(\lambda)^{-1}P_-^{-1}(\lambda). \end{aligned} \tag{2.59}$$

Using (2.55)–(2.57), we find the estimate for the jump matrix $W(\lambda)$ on γ_+,

$$W(\lambda) = I + \mathcal{O}(s_{-1}e^{2g}(\lambda - \lambda_0)\lambda_0^{-1/2}), \qquad \lambda \in \gamma_+, \tag{2.60}$$

Our next steps are similar to those presented in the proof of theorem 2.1. Consider the system of the singular integral equations for $Q_+(\lambda)$ equivalent to the RH problem (2.59), $Q_+ = I + \mathcal{K}Q_+$. Here, the singular integral operator $\mathcal{K}$ is the superposition of the multiplication operator in $W - I$ and of the Cauchy operator C_+. Because the Cauchy operator is bounded in $L_2(\gamma_+)$, the singular integral operator $\mathcal{K}$ for large enough $|x|$, $\arg x \in [(3\pi/5), \pi]$, satisfies the estimate

$$\|\mathcal{K}\|_{L_2(\gamma_+)} \leqslant c|s_{-1}||x|^{-1/2} \exp\left[-\tfrac{1}{5}2^{11/4}3^{1/4}|x|^{5/4}\cos\left(\tfrac{5}{4}(\arg x - \pi)\right)\right], \tag{2.61}$$

with some positive constant c whose precise value is not important for us. Thus equation $\zeta_+ = \mathcal{K}I + \mathcal{K}\zeta_+$ for the difference $\zeta_+ := Q_+ - I$ is solvable by iterations in the space $L_2(\gamma_+)$ for large enough $|x|$. The solution of the RH problem (2.59) is given by the integral $Q = I + \mathcal{K}I + \mathcal{K}\zeta_+$. This implies that the asymptotics of $Q(\lambda)$ as $\lambda \to \infty$,

$$\begin{aligned} Q(\lambda) &= I + \frac{1}{2\pi i}\int_{\gamma_+}(I + \mathcal{O}(\mathcal{K}I(\zeta)))(I - W^{-1}(\zeta))\frac{d\zeta}{\zeta - \lambda} \\ &= I + \mathcal{O}(\lambda^{-1}s_{-1}x^{-1/2}\exp(-\tfrac{1}{5}2^{11/4}3^{1/4}\mathrm{Re}(e^{-i\pi}x)^{5/4})). \end{aligned} \tag{2.62}$$

Now, let us find the asymptotics of the Painlevé function $y(x)$. Using (2.58), (2.57) and the estimate (2.62), we find that

$$\begin{aligned} X(\lambda) = I &+ \frac{s_{-1}}{\lambda\sqrt{\pi}}2^{-19/8}3^{-1/8}(e^{-i\pi}x)^{-1/8}\exp\left[-\tfrac{1}{5}2^{11/4}3^{1/4}(e^{-i\pi}x)^{5/4}\right](I + \mathcal{O}(x^{-3/8})) \\ &\times \begin{pmatrix} 1 & -2^{-1/4}3^{1/4}(e^{-i\pi}x)^{1/4} \\ 2^{1/4}3^{-1/4}(e^{-i\pi}x)^{-1/4} & -1 \end{pmatrix}. \end{aligned} \tag{2.63}$$

Comparing (2.63) and (2.50), we conclude that the Hamiltonian function for $s_0 = 0$ is

$$\mathcal{H}(x) = \mathcal{H}_0(x) - \frac{s_{-1}}{\sqrt{\pi}}2^{-17/8}3^{-3/8}(e^{-i\pi}x)^{-3/8}\exp\left[-\tfrac{1}{5}2^{11/4}3^{1/4}(e^{-i\pi}x)^{5/4}\right](1 + \mathcal{O}(x^{-1/8})), \tag{2.64}$$

while the Painlevé function is given by

$$y(x) = y_0(x) + \frac{s_{-1}}{\sqrt{\pi}}2^{-11/8}3^{-1/8}(e^{-i\pi}x)^{-1/8}\exp\left[-\tfrac{1}{5}2^{11/4}3^{1/4}(e^{-i\pi}x)^{5/4}\right](1 + \mathcal{O}(x^{-3/8})), \tag{2.65}$$

where $\mathcal{H}_0(x)$ and $y_0(x)$ are the Hamiltonian and the Painlevé functions, respectively, corresponding to $s_0 = s_{-1} = 0$. □

2.2. Other degenerate Painlevé functions

Applying the symmetry (2.16a) to the solution (2.54) and changing the argument of x in 2π, we obtain

Theorem 2.3. *If* $s_0 = 0$ *and* $|x| \to \infty$, $\arg x \in [\pi, (7\pi/5)]$, *then the asymptotics of the Painlevé first transcendent is given by*

$$y(x) = y_1(x) - \frac{s_1}{\sqrt{\pi}} 2^{-11/8} 3^{-1/8} (\mathrm{e}^{-\mathrm{i}\pi} x)^{-1/8} \exp[-\tfrac{1}{5} 2^{11/4} 3^{1/4} (\mathrm{e}^{-\mathrm{i}\pi} x)^{5/4}](1 + \mathcal{O}(x^{-3/8})), \tag{2.66}$$

where $y_1(x) \sim \sqrt{\mathrm{e}^{-\mathrm{i}\pi} x/6}$ *is the solution of the Painlevé equation for* $s_0 = s_1 = 0$, $s_{-1} = s_2 = s_{-2} = \mathrm{i}$.

The solutions $y_0(x)$ and $y_1(x) = \overline{y_0(\mathrm{e}^{2\pi \mathrm{i}} \bar{x})}$ are meromorphic functions of $x \in \mathbb{C}$ and thus can be continued beyond the sectors indicated in theorems 2.2 and 2.3. To find the asymptotics of $y_1(x)$ in the interior of the sector $\arg x \in [3\pi/5, \pi]$, we apply (2.54). Similarly, we find the asymptotics of the solution $y_0(x)$ in the interior of the sector $\arg x \in [\pi, 7\pi/5]$ using (2.66). Either expression implies

Corollary 2.4. *If* $|x| \to \infty$, $\arg x \in [3\pi/5, 7\pi/5]$, *then*

$$y_1(x) - y_0(x) = \frac{\mathrm{i}}{\sqrt{\pi}} 2^{-11/8} 3^{-1/8} (\mathrm{e}^{-\mathrm{i}\pi} x)^{-1/8} \exp[-\tfrac{1}{5} 2^{11/4} 3^{1/4} (\mathrm{e}^{-\mathrm{i}\pi} x)^{5/4}](1 + \mathcal{O}(x^{-3/8})). \tag{2.67}$$

Applying symmetries (2.16) to $y_k(x)$, $k = 0, 1$, we find the solutions $y_k(x)$ corresponding to the Stokes multipliers $s_k = s_{k-1} = 0$,

$$\begin{aligned} y_{2n}(x) &= \mathrm{e}^{\mathrm{i}(4\pi/5)n} y_0(\mathrm{e}^{\mathrm{i}(2\pi/5)n} x) \qquad &&\text{for } s_{2n} = s_{2n-1} = 0, \\ y_{2n+1}(x) &= \mathrm{e}^{\mathrm{i}(4\pi/5)n} y_1(\mathrm{e}^{\mathrm{i}(2\pi/5)n} x) \qquad &&\text{for } s_{2n+1} = s_{2n} = 0. \end{aligned} \tag{2.68}$$

Since there is one-to-one correspondence between the points of the monodromy surface and the Painlevé functions, the first of the identities (2.8′), $s_{k+5} = s_k$, implies that $y_{n+5}(x) = y_n(x)$.

Using theorems 2.2 and 2.3, we find that

$$\begin{aligned} y_{4n}(x) = y_{4n+5}(x) &= \sqrt{\mathrm{e}^{-\mathrm{i}\pi}(x/6)} + \mathcal{O}(x^{-2}), \\ &|x| \to \infty, \quad \arg x \in \left[\frac{\pi}{5} - \frac{4\pi}{5}n, \pi - \frac{4\pi}{5}n\right], \end{aligned} \tag{2.69}$$

$$\begin{aligned} y_{4n-2}(x) = y_{4n+3}(x) &= -\sqrt{\mathrm{e}^{-\mathrm{i}\pi}(x/6)} + \mathcal{O}(x^{-2}), \\ &|x| \to \infty, \quad \arg x \in \left[\frac{3\pi}{5} - \frac{4\pi}{5}n, \frac{7\pi}{5} - \frac{4\pi}{5}n\right]. \end{aligned} \tag{2.70}$$

The symmetry (2.16) with the definition (2.68) applied to (2.67) yields

Corollary 2.5. *If* $|x| \to \infty$ *and* $\arg x \in [(3\pi/5) - (2\pi/5)n, (7\pi/5) - (2\pi/5)n]$, *then*

$$\begin{aligned} y_{2n+1}(x) - y_{2n}(x) = \frac{1}{\sqrt{\pi}} \exp\left(\mathrm{i}\frac{\pi}{2} + \mathrm{i}\frac{4\pi}{5}n\right) 2^{-11/8} 3^{-1/8} \left[\exp\left(-\mathrm{i}\pi + \mathrm{i}\frac{2\pi}{5}n\right) x\right]^{-1/8} \\ \times \exp\left(-\tfrac{1}{5} 2^{11/4} 3^{1/4} \exp\left(-\mathrm{i}\pi + \mathrm{i}\frac{2\pi}{5} n x\right)\right)^{5/4} (1 + \mathcal{O}(x^{-3/8})). \end{aligned} \tag{2.71}$$

On the one hand, equations (2.67), (2.71) constitute the quasi-linear Stokes phenomenon for the Painlevé first equation. On the other hand, these equations give the asymptotic description of the degenerate Painlevé functions beyond the sectors in (2.69) and (2.70). Observing that the difference (2.71) is exponentially small in the interior of the indicated sector, we conclude that the asymptotics (2.69) and (2.70) as $|x| \to \infty$ continue to wider open sectors,

$$y_{4n}(x) = \sqrt{\mathrm{e}^{-\mathrm{i}\pi}\frac{x}{6}} + \mathcal{O}(x^{-2}), \qquad \arg x \in \left(\epsilon - \frac{\pi}{5} - \frac{4\pi}{5}n, \frac{7\pi}{5} - \frac{4\pi}{5}n - \epsilon\right), \tag{2.72}$$

$$y_{4n-2}(x) = -\sqrt{\mathrm{e}^{-\mathrm{i}\pi}\frac{x}{6}} + \mathcal{O}(x^{-2}), \qquad \arg x \in \left(\epsilon + \frac{\pi}{5} - \frac{4\pi}{5}n, \frac{9\pi}{5} - \frac{4\pi}{5}n - \epsilon\right), \tag{2.73}$$

where $\epsilon > 0$ is an arbitrary small constant.

Remark 2.3. The solutions $y_n(x)$ (2.68) corresponding to the trivial values of two Stokes multipliers $s_n = s_{n-1} = 0$ are the most degenerate among the Painlevé transcendents since they behave algebraically in four of the five sectors $\arg x \in (-\frac{\pi}{5}+\frac{2\pi}{5}k, \frac{\pi}{5}+\frac{2\pi}{5}k)$, $k = 0, \pm 1, \pm 2$, see (2.72), (2.73). Nevertheless, these solutions are transcendent, since their asymptotics as $|x| \to \infty$ within the remaining sector involves the elliptic function of Weierstrass, for more details see [30]. Moreover, the fact that the asymptotics of $y_n(x)$ is not elliptic in four sectors uniquely determines the values of all the Stokes multipliers s_k. Thus the asymptotics (2.72), (2.73) uniquely determine the degenerate solutions $y_n(x)$.

Remark 2.4. The asymptotics of less degenerate solutions corresponding to $s_n = 0$ and $s_{n+1} + s_{n-1} = \mathrm{i}$ can be found by applying the symmetries (2.16b) to equations (2.54) and (2.66).

3. Coefficient asymptotics

Using the steepest descent approach, cf [42], we can show the existence of the asymptotic expansion of $y_n(x)$, $n \in \mathbb{Z}$, in the negative degrees of $x^{1/2}$. Further elementary investigation of the recursion relation for the coefficients of the series allows us to claim that the asymptotic expansion for $y_n(x)$ in (2.72), (2.70) has the following form:

$$\begin{aligned} y_f(x) &= \sigma(-\tfrac{x}{6})^{1/2}\sum_{k=0}^{\infty} a_k\sigma^k(-x)^{-5k/2} + \mathcal{O}(x^{-\infty}) \\ &= \sigma(-\tfrac{x}{6})^{1/2}\sum_{k=0}^{\infty} a_{2k}(-x)^{-5k} + \tfrac{1}{\sqrt{6}}(-x)^{-2}\sum_{k=0}^{\infty} a_{2k+1}(-x)^{-5k} + \mathcal{O}(x^{-\infty}), \\ &\quad \sigma^2 = 1, \end{aligned} \tag{3.1}$$

where coefficients a_k are determined uniquely by the recurrence relation

$$a_0 = 1, \qquad a_{k+1} = \frac{25k^2 - 1}{8\sqrt{6}}a_k - \frac{1}{2}\sum_{m=1}^{k} a_m a_{k+1-m}. \tag{3.2}$$

Several initial terms of the expansion are given by

$$\begin{aligned} y_f(x) &= \sigma\sqrt{-x/6}\left\{1 + \frac{49}{768x^5} - \frac{4412\,401}{1179\,648x^{10}} + \frac{245\,229\,441\,961}{100\,663\,296x^{15}} + \mathcal{O}(x^{-20})\right\} \\ &\quad - \frac{1}{48x^2}\left\{1 - \frac{1225}{192x^5} + \frac{73\,560\,025}{49\,152x^{10}} - \frac{7759\,635\,184\,525}{3538\,944x^{15}} + \mathcal{O}(x^{-20})\right\}. \end{aligned} \tag{3.3}$$

Our next goal is to determine the asymptotics of the coefficients a_k in (3.1) as $k \to \infty$. With this purpose, let us construct a sectorial analytic function $\hat{y}(t)$,

$$\arg t \in [-\tfrac{2\pi}{5}(n+1), -\tfrac{2\pi}{5}n] : \quad \hat{y}(t) = y_{4n}(\mathrm{e}^{\mathrm{i}\pi}t^2), \qquad n = -2, -1, 0, 1, 2. \tag{3.4}$$

The function $\hat{y}(t)$ has a finite number of poles all contained in a circle $|t| < \rho$ and is characterized by the uniform asymptotic expansion near infinity,

$$\hat{y}(t) = \frac{t}{\sqrt{6}} \sum_{k=0}^{\infty} a_k t^{-5k} + \mathcal{O}(t^{-\infty}). \tag{3.5}$$

Let $y^{(N)}(t)$ be a partial sum

$$y^{(N)}(t) = \frac{t}{\sqrt{6}} \sum_{k=0}^{N-1} a_k t^{-5k}, \tag{3.6}$$

and $v^{(N)}(t)$ be a product

$$v^{(N)}(t) = t^{5N-2}\sqrt{6}(\hat{y}(t) - y^{(N)}(t)) = t^{-1} \sum_{k=0}^{\infty} a_{k+N} t^{-5k} + \mathcal{O}(t^{-\infty}). \tag{3.7}$$

Because $t^{5N-2}y^{(N)}(t)$ is a polynomial, the integral of $v^{(N)}(t)$ along the counter-clockwise oriented circle of radius $|t| = \rho$ satisfies the estimate

$$\left| \oint_{|t|=\rho} v^{(N)}(t)\,\mathrm{d}t \right| \leqslant \rho^{5N-2}\sqrt{6} \oint_{|t|=\rho} |\hat{y}(t)|\,\mathrm{d}l \leqslant \sqrt{6}\, 2\pi\rho^{5N-1} \max_{|t|=\rho} |\hat{y}(t)| = C\rho^{5N} \tag{3.8}$$

with some positive constant C whose precise value is not important for us.

On the other hand, inflating the sectorial arcs of the circle $|t| = \rho$, we find that

$$\oint_{|t|=\rho} v^{(N)}(t)\,\mathrm{d}t = \oint_{|t|=R} v^{(N)}(t)\,\mathrm{d}t + \sum_{n=-2}^{2} \int_{\exp[\mathrm{i}(2\pi/5)n](\rho,R)} \left(v_+^{(N)}(t) - v_-^{(N)}(t) \right) \mathrm{d}t. \tag{3.9}$$

Because $v^{(N)}(t) = t^{-1}a_N + \mathcal{O}(t^{-6})$, the first of the integrals in the rhs of (3.9) is computed as follows:

$$\oint_{|t|=R} v^{(N)}(t)\,\mathrm{d}t = 2\pi\mathrm{i}a_N + \mathcal{O}(R^{-5}). \tag{3.10}$$

The remaining integrals in (3.9) are computed using definitions (3.4)–(3.7) and (2.68) with the identification $y_{-4}(x) = y_1(x)$ and the formula (2.67) together,

$$\begin{aligned}
\sum_{n=-2}^{2} \int_{\exp[\mathrm{i}(2\pi/5)n](\rho,R)} \left(v_+^{(N)}(t) - v_-^{(N)}(t) \right) \mathrm{d}t &= 5\sqrt{6} \int_{(\rho,R)} t^{5N-2}(y_{-4}(\mathrm{e}^{\mathrm{i}\pi}t^2) - y_0(\mathrm{e}^{\mathrm{i}\pi}t^2))\,\mathrm{d}t \\
&= \mathrm{i}\frac{5\sqrt{6}}{\sqrt{\pi}} 2^{-11/8}3^{-1/8} \int_{(\rho,R)} t^{5N-\frac{9}{4}} \exp(-\tfrac{1}{5}2^{11/4}3^{1/4}t^{5/2})(1 + \mathcal{O}(t^{-3/4}))\mathrm{d}t \\
&= 2\mathrm{i}\frac{\sqrt{6}}{\sqrt{5}\sqrt{\pi}} (\tfrac{1}{5}2^{11/4}3^{1/4})^{-2N}\Gamma(2N - \tfrac{1}{2})(1 + \mathcal{O}(N^{-3/10})) + \mathcal{O}(\rho^{5N-(5/2)}) \\
&\quad + \mathcal{O}(\exp(-\tfrac{1}{5}2^{11/4}3^{1/4}R^{5/2})R^{5N-(15/4)}).
\end{aligned} \tag{3.11}$$

Thus, letting $R = \infty$, we find the asymptotics of the coefficient a_N in (3.1) as $N \to \infty$,

$$a_N = -\frac{\sqrt{6}}{\sqrt{5}\,\pi^{3/2}}\left(\tfrac{1}{5}2^{11/4}3^{1/4}\right)^{-2N}\Gamma(2N - \tfrac{1}{2})(1 + \mathcal{O}(N^{-3/10})) + \mathcal{O}(\rho^{5N}), \qquad N \to \infty. \tag{3.12}$$

Remark 3.1. The presented asymptotic formula shows a remarkable accuracy: neglecting error terms in (3.12), we find an approximation to a_N with the relative error not exceeding 2% for $N = 4$ and 1% for $N = 7$. Furthermore, for the initial set of $N = 1, 2, \ldots, 7$, the relative error decreases approximately as N^{-1}, which is significantly better than estimated.

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